1.2 Continuity of Probability

A sequence of events $\{A_n,\ n \geq 1\}$ is said to be *increasing sequence* if

$$A_n \subseteq A_{n+1},\ n \geq 1$$

and is said to be *decreasing* if

$$A_n \supseteq A_{n+1},\ n \geq 1.$$

Define:

1. the limit of increasing sequence

$$\lim_{n\to\infty} A_n = \bigcup_{n=1}^{\infty} A_n,$$

2. and similarly the limit of decresing sequence

$$\lim_{n\to\infty} A_n = \bigcap_{n=1}^{\infty} A_n.$$

Proposition. *If* $\{A_n,\ n \geq 1\}$ *is either increasing or decreasing sequence of events, then*

$$\lim_{n\to\infty} P(A_n) = P(\lim_{n\to\infty} A_n).$$

Proof is given in [1], pp. 2-3.

Example. *Somebody is tossing a perfect coin until either HEAD or TAIL appears twice. Find the probability that the number of tosses will be even.*

SOLUTION: Define

$$A_n = \{\text{the number of tosses is equal } = 2n\}$$

then

$$A_n = (\underbrace{T\,H \ldots T\,H}_{2(n-1)-times}\ T\,T) \cup (\underbrace{H\,T \ldots H\,T}_{2(n-1)-times}\ H\,H)$$

and

$$P(A_n) = \frac{1}{2^{2n-1}}.$$

The event

$$A = \{\text{the number of tosses is even}\} = \bigcup_{n=1}^{\infty} A_n = \lim_{N\to\infty} \bigcup_{n=1}^{N} A_n = B_N.$$

Then

$$P(A) = \sum_{n=1}^{\infty} P(A_n) = \sum_{n=1}^{\infty} \frac{1}{2^{2n-1}} = \frac{2}{3}.$$

$\square$